§3 Finite and Infinite Sets

(In this chapter, we use $\mathbb{N}^+$ to denote the set of all positive integers, i.e. $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.)

Definition 3.1

- The empty set is said to have 0 element
- A set S is said to have n elements if there exists a bijection from $\mathbb{N}_n = \{1, 2, \cdots, n\}$ onto S (denoted by $|S| = n$).
- A set S is said to be finite if it is either empty or it has n elements for some $n \in \mathbb{N}^+$.
- A set S is said to be infinite if it is NOT finite.

Definition 3.2

Let A and B be two sets.

- A and B have the same cardinality if there exists a bijection from A to B. It is denoted by $|A| = |B|$.
- A has cardinality less than or equal to the cardinality of B if there exists an injection from A to B. It is denoted by $|A| \le |B|$.
- A has cardinality less than the cardinality of B if there exists an injection, but no bijection, from A to B. It is denoted by $|A| < |B|$.

Lemma 3.1 (Pigeonhole Principle)

Let $m, n \in \mathbb{N}^+$ with $m > n$. Then there does not exist an injection from $\mathbb{N}_m$ into $\mathbb{N}_n$.

proof:

Induction on n.

There are 10 pots but only 9 covers

At least 2 pots share the same cover.

Proposition 3.1

If S is a finite set, then the number of elements of S is unique.

proof:

Claim: If $|S| = m$ and $|S| = n$, then $m = n$.

Suppose $|S| = m$ and $|S| = n$.

There exist bijections $f : \mathbb{N}_m \to S$ and $g : \mathbb{N}_n \to S$

Then $g^{-1} \circ f : \mathbb{N}_m \to \mathbb{N}_n$ is a bijection, by the above lemma $m \le n$.

Similarly, $f^{-1} \circ g : \mathbb{N}_n \to \mathbb{N}_m$ is a bijection and so $m \ge n$.

$\therefore m = n$.

Lemma 3.2

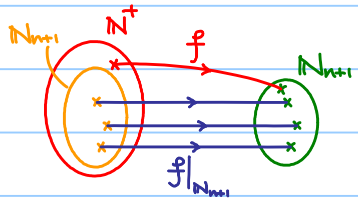
If $n \in \mathbb{N}^+$, there does not exist an injection from $\mathbb{N}^+$ to $\mathbb{N}_n$.

proof:

Note $\mathbb{N}_{n+1} \subseteq \mathbb{N}^+$, if there exists an injection $f : \mathbb{N}^+ \to \mathbb{N}_n$,

then the restriction $f|_{\mathbb{N}_{n+1}}$ is also an injection

from $\mathbb{N}_{n+1}$ into $\mathbb{N}_n$ (Contradiction).



Direct consequence of the above lemma:

Theorem 3.1

$\mathbb{N}^+$ is an infinite set.

Proposition 3.2

- If $|A| = m$, $|B| = n$ and $A \cap B = \phi$, then $|A \cup B| = m + n$.
- If $|A| = m$, $|C| = 1$ and $C \subseteq A$, then $|A \setminus C| = m - 1$.
- If C is infinite and B is finite, then $C \setminus B$ is infinite

proof: (Exercise)

Proposition 3.3

Suppose $T \subseteq S$.

- If S is finite, then T is finite.
- If T is infinite, then S is infinite.

Definition 3.3

- A set S is said to be countably infinite if there exists a bijection of $\mathbb{N}^+$ onto S.
- A set S is said to be countable if it is either finite or countably infinite.
- A set S is said to be uncountable if it is NOT countable.

Example 3.1

1) E = the set of all positive even number is countably infinite.

   Consider $f: \mathbb{N}^+ \to E$ defined by $f(n) = 2n$.

2) $\mathbb{Z}$ is countably infinite.

   How to construct a bijection from $\mathbb{N}^+$ onto $\mathbb{Z}$?

   Hint: $1 \mapsto 0,\ 2 \mapsto 1,\ 3 \mapsto -1,\ 4 \mapsto 2,\ 5 \mapsto -2$

   Idea: Construction an algorithm to go through all elements in $\mathbb{Z}$ one by one.

   (Exercise. Write down the function explicitly.)

Exercise 3.1

Prove that

a) If A and B are both countably infinite and $A \cap B = \phi$, then $A \cup B$ is also countably infinite.

b) If A and B are both countably infinite, then $A \cup B$ is also countably infinite.

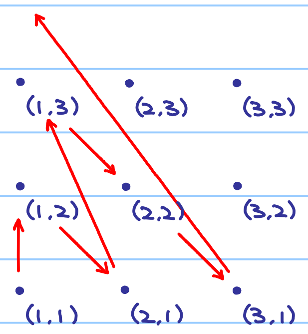
   (Using (a), $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$)

Proposition 3.4

$\mathbb{N}^+ \times \mathbb{N}^+$ is countably infinite.

Idea of the proof:

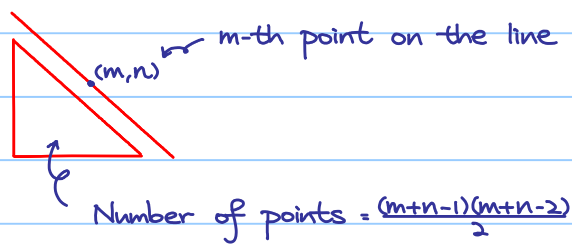
Go through every element in $\mathbb{N}^+ \times \mathbb{N}^+$ one by one.



Define $f: \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ by $f(m,n) = \frac{(m+n-1)(m+n-2)}{2} + m$

Exercise: Show that f is bijective.

(and so $f^{-1}: \mathbb{N}^+ \longrightarrow \mathbb{N}^+ \times \mathbb{N}^+$ is bijective.)



Proposition 3.5

Suppose $T \subseteq S$.

- If S is countable, then T is countable.
- If T is uncountable, then S is uncountable.

Proposition 3.6

The followings are equivalent (TFAE):

(a) S is countable

(b) There exists a surjection of $\mathbb{N}^+$ onto S.

(c) There exists an injection of S onto $\mathbb{N}^+$.

Theorem 3.2

$\mathbb{Q}^+$ is countably infinite.

Idea of proof:

- $f : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{Q}^+$ defined by $f(m,n) = \frac{m}{n}$ is a surjection.
- $\mathbb{N}^+ \times \mathbb{N}^+$ is countably infinite, i.e. there exists a bijection $g : \mathbb{N}^+ \to \mathbb{N}^+ \times \mathbb{N}^+$.

$\therefore$ $f \circ g : \mathbb{N}^+ \to \mathbb{Q}^+$ is a surjection.

$\mathbb{Q}^+$ is countable (by the previous theorem)

Furthermore, $\mathbb{N}^+ \subseteq \mathbb{Q}^+$ (By regarding $n = \frac{n}{1}$) which is infinite

$\therefore$ $\mathbb{Q}^+$ is infinite and so it can only be countably infinite.

$\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$, hence $\mathbb{Q}$ is also countably infinite.

Proposition 3.7

If A_m is a countable set for each $m \in \mathbb{N}^+$, then $A := \bigcup_{m=1}^{\infty} A_m$ is countable.

Troubles: 1) Some A_i's are finite while some A_j's are infinite

2) $A_i \cap A_j$ may NOT be empty.

proof :

For each $m \in \mathbb{N}^+$, let $\varphi_m : \mathbb{N}^+ \longrightarrow A_m$ be a surjection.

Then, define $f : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow A$ by $f(m,n) = \varphi_m(n)$.

Check f is a surjection.

Furthermore, $\mathbb{N}^+ \times \mathbb{N}^+$ is countably infinite.

Then, there exists a bijection $g : \mathbb{N}^+ \longrightarrow \mathbb{N}^+ \times \mathbb{N}^+$.

$\therefore$ $f \circ g : \mathbb{N}^+ \to A$ is a surjection and the result follows.

Theorem 3.3 (Cantor's Theorem)

If A is any set, then there exists no surjection of A onto $P(A)$, where $P(A)$ is the set of all subsets of A.

Think: If $A = \{1,2\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$.

Sort of clear if A is finite.

proof :

If $A = \phi$, the statement is trivial (If $A = \phi$, then $P(A) = \{\phi\}$).

Assume A is nonempty.

Suppose $\psi : A \to P(A)$ is a surjection.

Then pick an element $a \in A$, $\psi(A)$ is a subset of A, we either have $a \in \psi(a)$ or $a \notin \psi(a)$.

Let $D = \{a \in A : a \notin \psi(a)\}$ which is again a subset of A.

By surjectivity of ψ, $D = \psi(a_0)$ for some a_0

Now, $a_0 \in D$ or $a_0 \notin D$?

However, both cases give contradiction!

Consequences :

1) $|A| < |P(A)| < |P(P(A))| < \cdots$ ascending sequence.

2) There exist no surjection from $\mathbb{N}^+$ onto $P(\mathbb{N}^+)$

$\therefore$ $P(\mathbb{N}^+)$ is uncountable (Existence of uncountable set)